

Are Avalanches in Sandpiles a Chaotic Phenomenon?¹

Maria de Sousa Vieira²

International Center for Complex Systems, Departamento de Fisica, UFRN, 59072-970 Natal, RN, Brazil.

Abstract

We show that deterministic systems with strong nonlinearities seem to be more appropriate to model sandpiles than stochastic systems or deterministic systems in which discontinuities are the only nonlinearity. In particular, we are able to reproduce the breakdown of Self-Organized Criticality found in two well known experiments, that is, a centrally fueled pile [Held *et al. Phys. Rev. Lett.* **65** (1990) 1120] and sand in a rotating tray [Bretz *et al. Phys. Rev. Lett.* **69** (1992) 2431]. By varying the parameters of the model we recover Self-Organized Criticality, in agreement with other experiments and other models. We show that chaos plays a fundamental role in the dynamics of the system.

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In 1987 Bak, Tang and Wiesenfeld showed that certain open, dissipative, spatially extended systems spontaneously achieve a critical state characterized by power-law distribution of event sizes(1). They denoted this phenomenon Self-Organized Criticality (SOC) and illustrated it using a simple two-dimensional cellular automaton model for sandpiles. An enormous amount of research followed that work. Some natural phenomena that have been connect to SOC are earthquakes, evolution, interface dynamics, solar flares, vortices in superconductors, charge density waves among others. For a review see(2; 3).

The first observation of SOC in granular material was made by Frette *et al.*(4). They observed that elongated grains of rice dropped between two parallel plates with a narrow separation between them (that is, in a quasi-one-dimensional geometry) present avalanche distribution consistent with SOC. Using more symmetric rice a stretched exponential was seen.

¹ In honor of Per Bak.

² e-mail: mariav_us@yahoo.com.

For systems in which the flow of grains is on a two dimensional surface different behaviors have been reported. To our knowledge, the first of such experiments was performed by Jaeger *et al.*(5). They showed that avalanches of sand in a rotating drum do not present power-law distribution of event sizes, instead they found system-spanning avalanches with a narrowly peaked distribution. Similar results were found by them for a avalanches of sand in a box with an open side(5). Using a different set up, Held *et al.*(6) studied avalanche distribution by fueling a canonical pile on its apex. They observed that if the base of the pile is smaller than a given size, then the avalanche distribution is consistent with SOC. If the base of the pile is increased beyond that critical value, another regime dominated by big, nearly periodic avalanches is observed. Bretz *et al.* performed experiments with granular material in a rotating tray(7). They found big spanning avalanches that reset the system, but between those large avalanches a sequence of small avalanches was found which presents power-law distribution.

New experiments on granular materials with flow in two dimensions have shown avalanche distribution fully consistent with SOC. The main difference of these experiments with respect to the previous ones was a change in the grain type. Costello *et al.* studied the avalanche distribution in a canonical bead pile(8) whereas Aegerter *et al.* studied a rice pile in a box where grains are dropped at one of its edge(9). In both experiments power-law distribution of event size and finite size scaling data collapse was seen.

Recently, two simple nonlinear two-dimensional deterministic models governed by coupled maps lattice (CML) have been introduced by us which reproduce the avalanche distribution observed in the experiments with granular materials(10). We modeled the experiments in which a canonical pile is fueled at its apex and the experiment in which the granular material is in a rotating tray. We observed that in both models there is a region of the parameter space where SOC occurs, and another region where SOC breaks down. The way SOC breaks is different from one model to the other and it is as in the experiments.

Here we expand that work to show that the universality class of those systems is not unique. That is, in each model it changes with the parameter values. We also study the Liapunov exponent of the models and show that at the onset of exponential instability the distribution of event sizes crosses from a power-law one to one dominated by big avalanches. Strong chaos is therefore a deciding factor for SOC to be present or not, and is closely related to the amount of friction between the grains.

For the sake of completeness we review the algorithms for evolving our systems(10). We first describe the one that governs the centrally fueled pile, called in (10) “local dropping” (LD): (1) Start the system by assigning random initial val-

ues for the variables $x_{i,j}$, where $x_{i,j}$ is the local slope of the pile, so the they are below a chosen, fixed, threshold x_{th} . (2) Choose a nearly central site of the lattice and update it slope according to $x_{i,j} = x_{th}$. (3) Check the slope in each element. If an element i, j has $x_{i,j} \geq x_{th}$, update $x_{i,j}$ according to $x'_{i,j} = \phi(x_{i,j} - x_{th})$, where ϕ is a given nonlinear function that has two parameters a and d . Increase the slope in all its nearest neighboring element according to $x'_{nn} = x_{nn} + \Delta x/4$, where $\Delta x = x_{i,j} - x'_{i,j}$ and nn denotes nearest neighbors. (4) If $x'_{i,j} < x_{th}$ for all the elements, go to step (2) (the event, or avalanche, has finished). Otherwise, go to step (3) (the event is still evolving). Without losing generality, we can take $x_{th} = 1$. In our simulation in step (2) we have chosen the site with $i = j = L/2$, where L is the lattice size. The nonlinear function we use is

$$\phi(x) = \begin{cases} 1 - d - ax, & \text{if } x < (1 - d)/a, \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

The parameter d is in the interval $(0; 1]$ and is associated with the minimum drop in energy after an event involving one single element. The parameter a is greater or equal to zero and is associated with the amount of dynamic friction between the grains. That is, the smaller the a , the larger the friction and the smaller the change in the slope of the pile.

The algorithm that governs the model used for rotating tray experiment, that is, the “uniform driving case” (UD), is similar to the one given above, with the exception of step (2). That step is replaced by: (2) Find the element in the lattice that has the largest x denoted here by x_{max} . Then update all the lattice elements according to $x_{i,j} \rightarrow x_{i,j} + x_{th} - x_{max}$.

As one evolves the system one can study several quantities, such as avalanche size (which is the number of site updates in a avalanche), time duration, radius of gyration, etc. Here we limit our study to the avalanche size distribution. We find that in both models, when $d = 1$ the only nonlinearity in the system is a discontinuity between x_{th} and $\phi(0)$ and we observed that in this case SOC is always present. Note that when $d = 1$ the UD case reduces to the conservative OFC model, which is known to present SOC. When $a \leq 1$ SOC is found for any value of d , whereas when $a > 1$ and $d < 1$ scaling consistent with SOC happens only for L below a given value. For L larger than that critical size, big avalanches that belong to a different distribution are observed. This is consistent with what is found in experiments with granular materials. In Fig.1 we show the probability distribution of avalanche sizes $P(s)$ as a function of it size s for a regime with small d and $a > 1$ ($d = 0.1$ and $a = 1.2$). In (a) we show the case of local dropping for L ranging from 8 to 256. In (b) we show similar distribution for the uniform driving case for the same system sizes. One sees that for small systems sizes ($L \leq 32$ in the LD case and $L \leq 16$ in the UD case) a scaling regime consistent with SOC is seen. We have found finite size scaling

Fig. 1. Distribution of event sizes for $a = 1.2$, $d = 0.1$, $L = 8, 16, 32, 64, 128, 256$ for the (a) LD and (b) UD cases.

data collapse for both cases and for the LD case it is reported in (10). This scaling regime reproduces what is observed experimentally in the pile fueled at its apex(6). As the system size is increased a new regime is seen where the SOC scaling is lost by the appearance of frequent large avalanches, consistent with experiments. Note that for the regime of broken SOC the distribution of the small events do not change as the system size increases in the LD case, but the the UD case the small avalanches become less frequent as the system size increases. This could explain why in some experiments, such as (5) modeled by the UD case only large avalanches was seen. We have also found that the big avalanches in the UD case involves all the elements of the system, and this does not necessarily occurs in the LD case.

We now turn our attention to the regime when $a < 1$, where the system presents SOC for any value of d and L . We plot in Fig.2(a) the avalanche distribution for the LD case for two different values of d , that is, $d = 0.1$ (solid line) and $d = 1$ (dashed line) with $a = 0.9$. One sees that $P(s) \sim s^{-\tau}$, characteristic of SOC systems but the value of τ is different from one case to the other. We have varied the system size and used finite size scaling ansatz by plotting $P(s)s^\tau = s/L^D$ for $L = 64, 128, 256$. The results are displayed in Fig.2(b). The lower curves are for $d = 1$ and the upper ones for $d = 0.1$. We observe that finite size scaling is reasonably well obeyed except at the region where the curves bend for $d = 0.1$. We have used $\tau = 1.26$ and $D = 2.6$ for $d = 0.1$ and $\tau = 1.33$ and $D = 2.8$ for $d = 1$. We see that the critical exponents τ and D vary as we vary the parameters. Therefore, the universality class of this system is not unique. In Fig.2(c) we show similar results for the UD case. The solid line is for $d = 0.1$ and the dashed one for $d = 1$. In Fig.2(d) we show the data collapse using the finite size scaling ansatz for $L = 64, 128, 256$. The curves with a smoother bending are for $d = 1$. We have used $\tau = 1.33$ and $D = 2.7$ for $d = 0.1$ and $\tau = 1.27$ and $D = 3.0$ for $d = 1$. Again, we see a change of universality class when the parameters of the system are varied.

We have studied the largest Liapunov exponent (LLE) of the system to verify the presence or absence of chaos. If the LLE is positive, the system is said to be chaotic. To study the LLE we use the algorithm developed by Benettin *et al.*(11) and the results are displayed in Fig.3. In (a) and (b) we show the LLE as a function of a for the LD and UD cases, respectively. There we have $d = 0.1$,

Fig. 2. Distribution of event sizes for (a) $a = 0.9$, $L = 256$ with $d = 0.1$ (solid) and $d = 1$ (dashed) for the LD case. (b) Finite-size scaling data collapse for $a = 0.9$, $L = 64, 128, 256$ with $d = 0.1$ (upper set of curves) and $d = 1$ (lower set of curves) for the LD case. (c) and (d) Same as (a) and (b), respectively, except that now it applies for the UD case.

$L = 32$ (dashed) and $L = 64$ (solid). The figures show that the LLE is positive for any value of a and that at $a = 1$ a transition happens. If no discontinuities existed between the x_{th} and $\phi(0)$ two trajectories with slightly different initial conditions would tend to converge when $a < 1$ (see Ref.(10; 12)) resulting in a negative LLE. However, because of that discontinuity two competing tendencies exist: one for convergence and another for separation. However, the one for separation is stronger and this results in a positive LLE, as Fig.3 shows. This is a weak form of chaos. When $a > 1$ there are no competing tendencies, and the trajectories always separate, and this will obviously result in a positive LLE. This is a strong form of chaos. The transition $a < 1$ (where SOC is found) to the one in which $a > 1$ (where SOC is broken for large L) is markedly different in the two models.

How the LLE vary with the system size is shown in Fig.3(c) for the LD case and Fig.3(d) for the UD case. The dashed line refers to $a = 1.5$ and the solid one to $a = 0.5$. There, $d = 0.1$. One sees that for small L in both curve the LLE tends to decreases. However, as L grows the case in which $a < 1$ (where SOC is present for any L) the LLE continues to get smaller and goes to zero (as a power-law function for the LD case), whereas when $a > 1$ the curves change their behavior as L gets bigger (which is consistent with the fact that SOC breaks for a given value of L when $a > 1$). That is, as the system size increases the chaotic behavior of the system decreases rapidly in the SOC regime, whereas this does not occur in the non SOC regime (in this regime the LLE tends to increase or stays approximately constant).

We summarize our results as follows: the introduction of strong nonlinearities in SOC systems causes the appearance of a new behavior as the parameters are changed. This is not seen in SOC systems in which discontinuities are the only nonlinearity. That is, one is able to see the breakdown of SOC as the system size increases, which is seen in experiments. Chaos play an important role, since it is only in the regime of strong chaos that the breakdown of SOC occurs.

Fig. 3. The largest Liapunov exponent as a function of a for $d = 0.1$, $L = 32$ (dashed) and $L = 64$ (solid) for (a) LD and (b) UD cases. The largest Liapunov exponent as a function of L for $a = 0.5$ (solid) and $a = 1.5$ (dashed) for (c) UD and (d) UD cases.

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Fig. 1

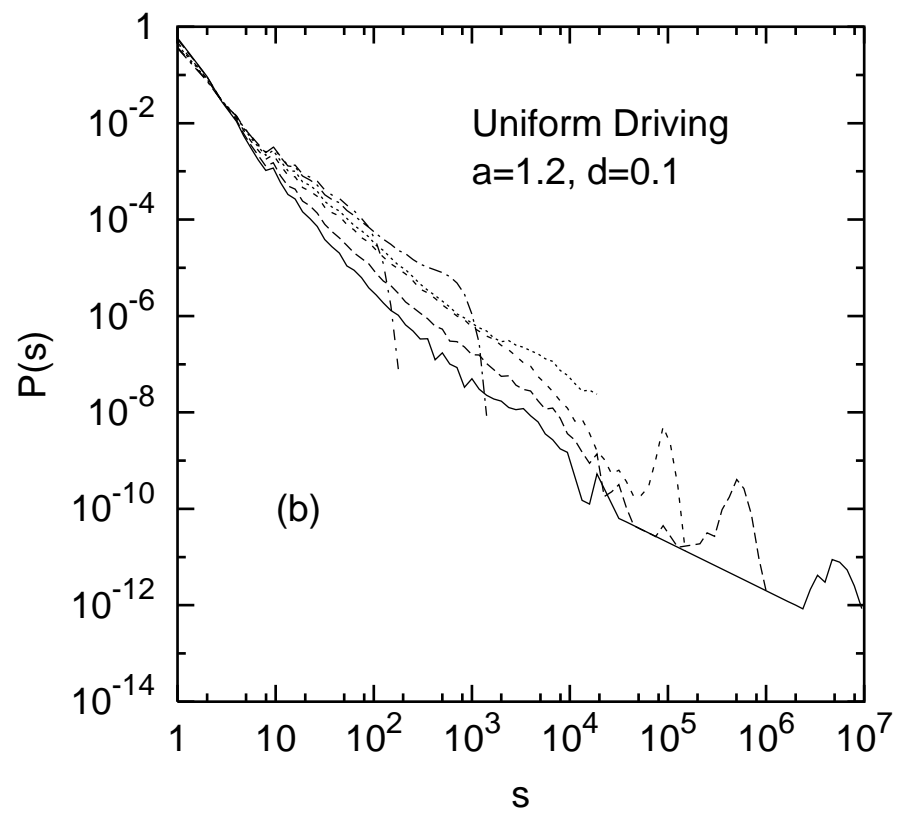
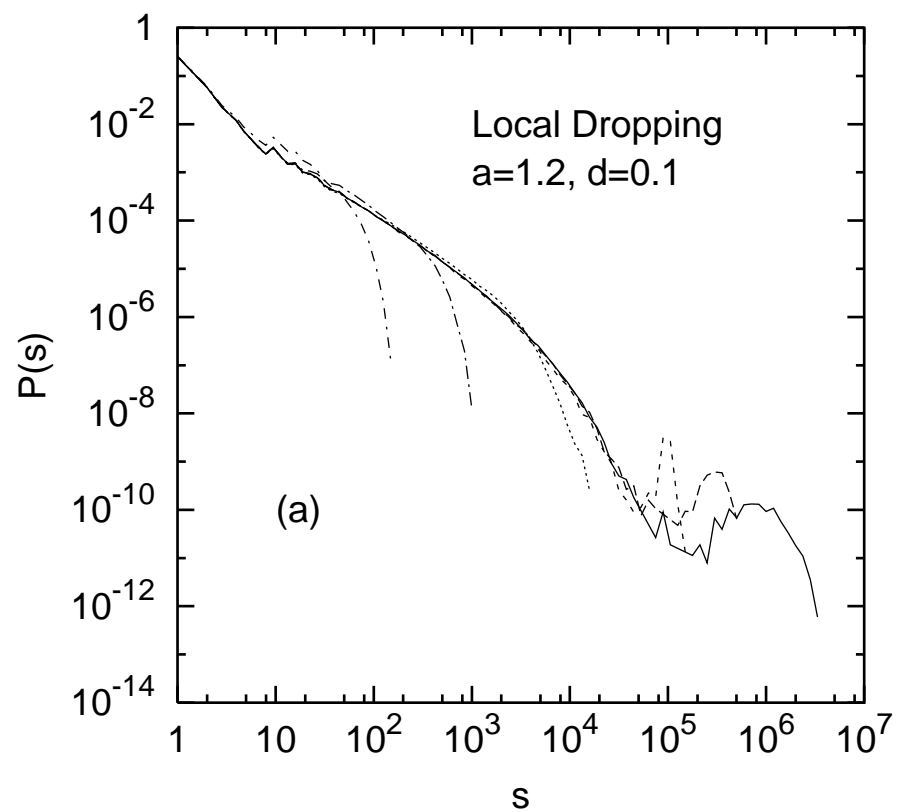


Fig. 2

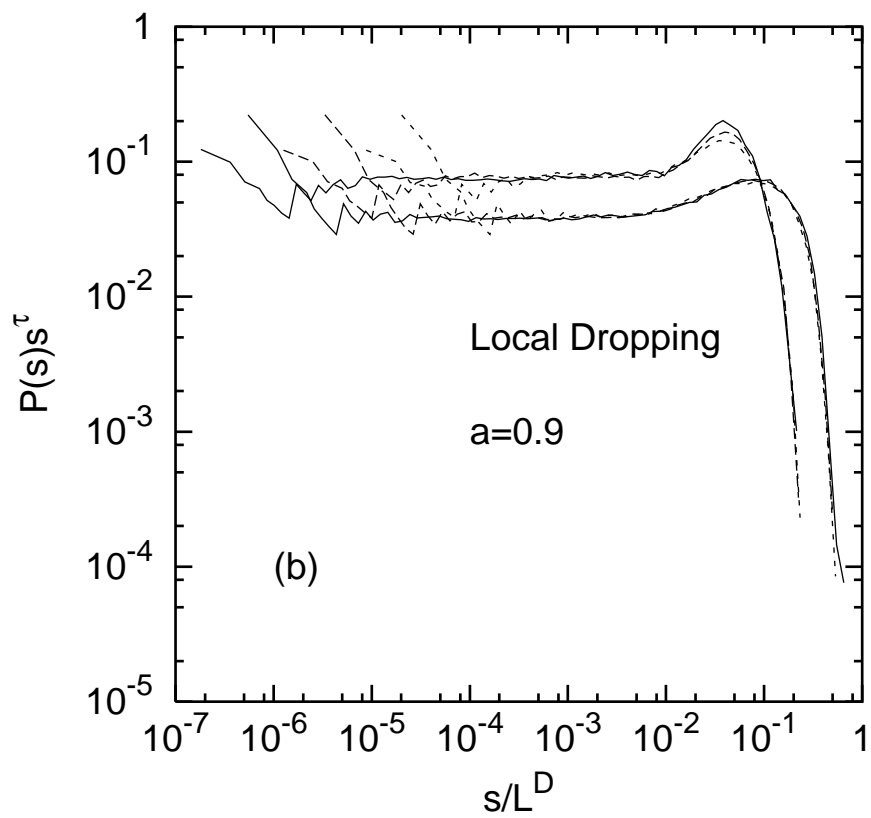
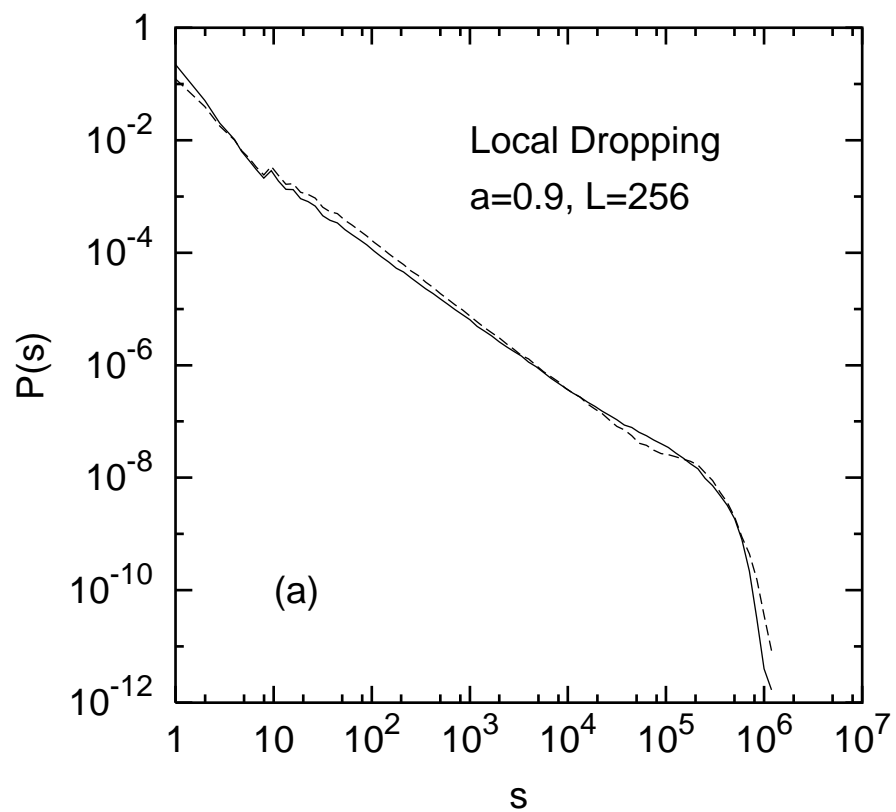


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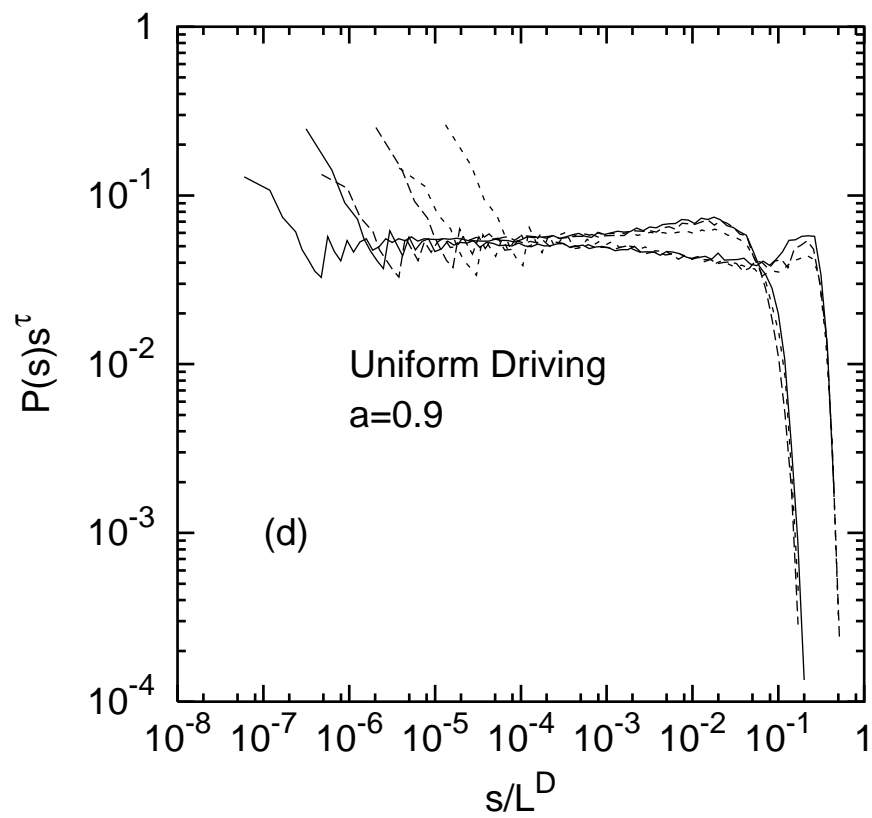
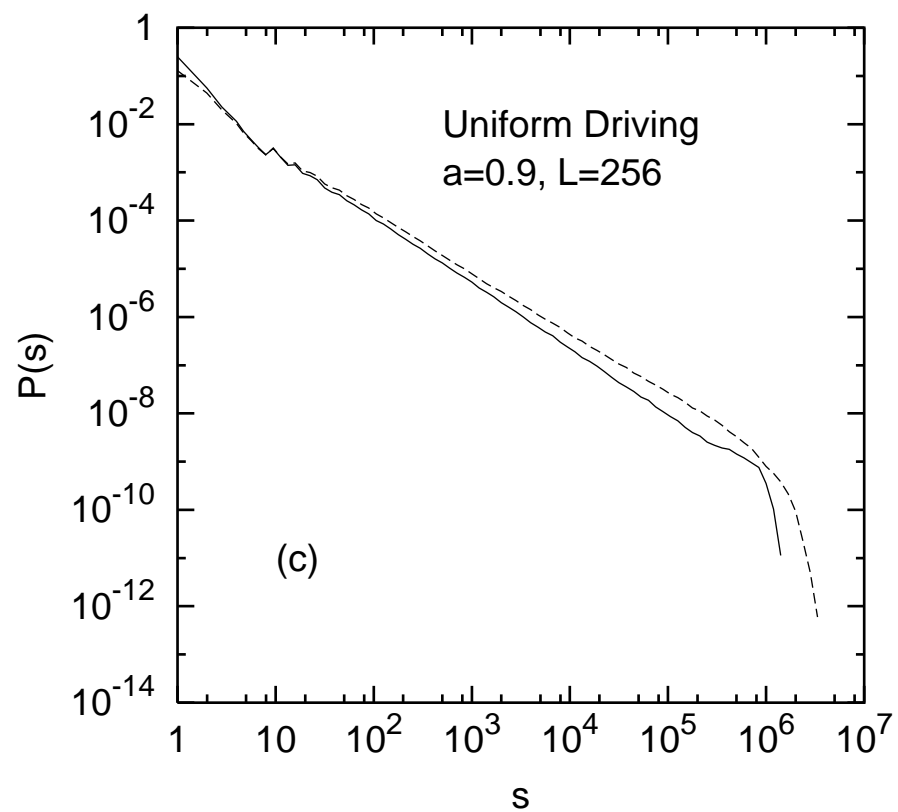


Fig. 3

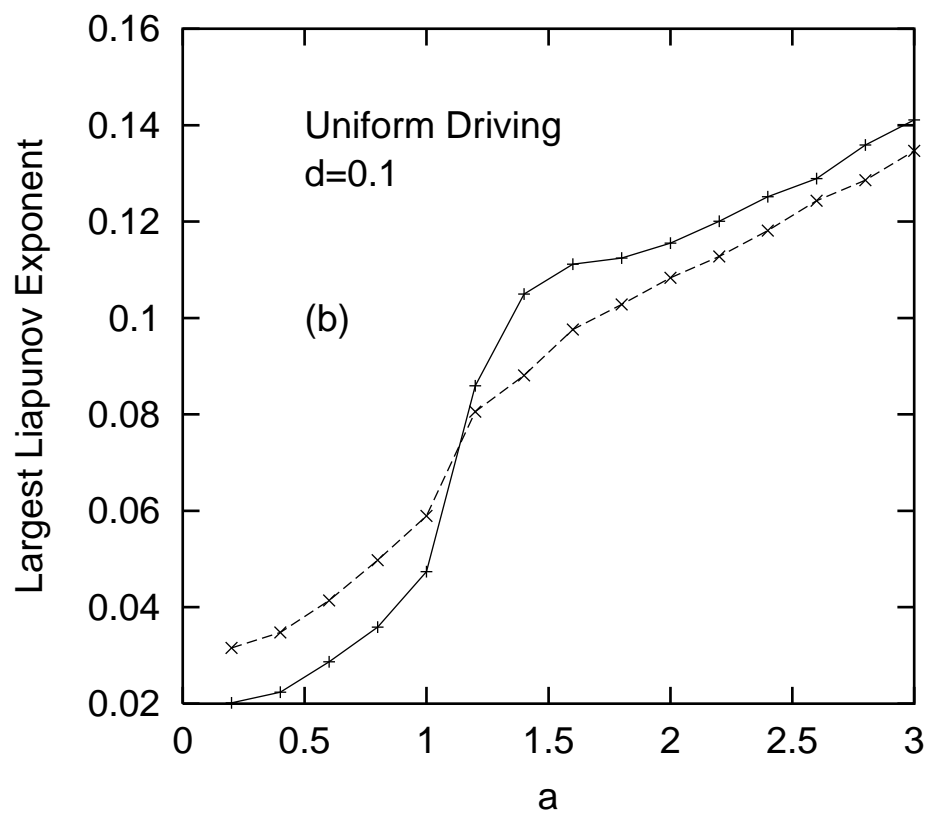
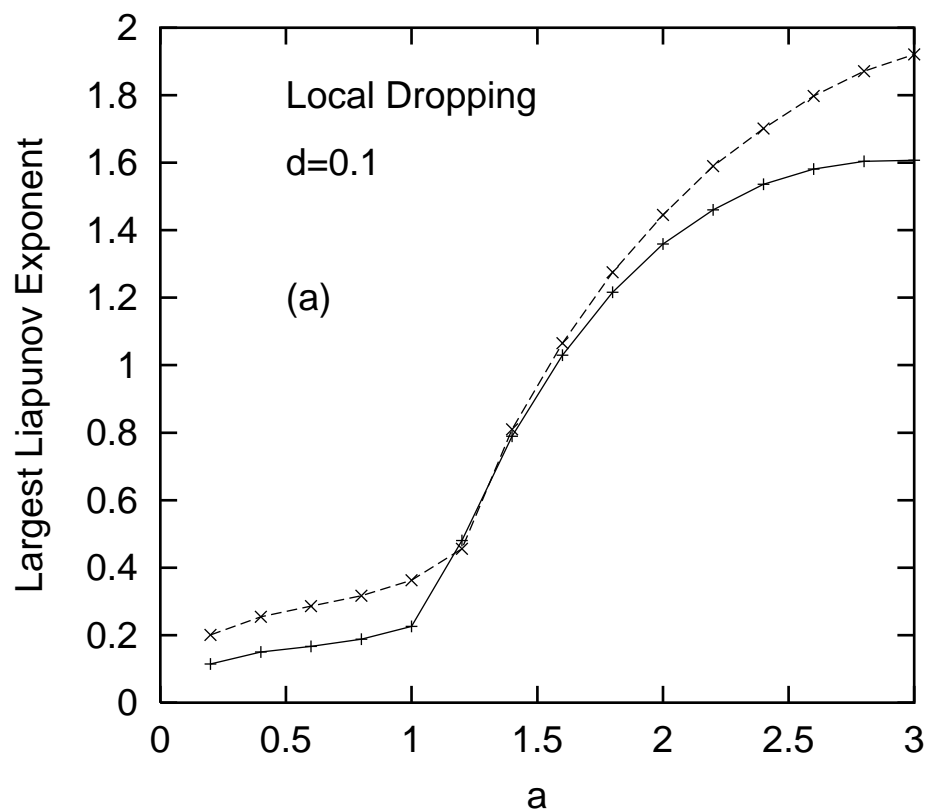


Fig. 3

